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Solution of polyhedra

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Abstract. The paper is an exposition of the author's talk on the Seminar on Differential Geometry in IMPA in Rio de Janeiro. It presents a short survey of some recent results in the metric theory of polyhedra in 3-space. Namely we emphasize on some applicatons of the theorem which is a vast generalization of the Heron's formule for the area of a triangle to volumes of polyhedra.

Keywords: polyhedral metric, isometric realization, polyhedron, algorithm, volume, diagonals.

Mathematical subject classification: 51N25, 52B10, 52C25.

1 Introduction: a bit of history

The origin of results about which I am going to talk today is in the theory of flexions of polyhedra. *A flexion* of a polyhedron P is a continuous deformation during which each face of P remains congruent to itself (we mean that all considerations are in three-dimensional euclidean space). A *trivial* flexion is a motion of P in the space meanwhile *a nontrivial* flexion changes at least one dihedral angle of P. If a polyhedron does not admit any nontrivial flexion it is called *continuously rigid*. The first remarkable result in this theory goes back to Cauchy who proved in 1813 [1] that any convex polyhedron is continuously rigid. In reality he affirmed that any convex polyhedron is globally rigid (this means that if two convex polyhedra P_1 and P_2 have the same combinatorial structure and congruent corresponding faces then they are congruent or symmetric). But the Cauchy's proof of both his famous lemmas contained gapes one of which was corrected by Lebesgue [2] almost 100 years later and the second one was eliminated still later in [4][1]. During a long period of time there was no any example

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[1] Note that this theorem was conjectured by Legendre in 1794 in the first edition of his ''Eléments de géométrie'' [3] and, moreover, in the same book he formulated and proved the lemma about

of a flexible polyhedron. Only in 1897 [5] Bricard discovered the existence of flexible octahedra and gave their complete classification. He proved that among flexible octahedra there are three types distinguished one from other by some relations between the lengths of edges and disposition in the space.

For example, a flexible octahedron of the 1-st type has the equator $ABCD$ with equal opposite edges $AB = CD$, $AD = BC$ (Fig. 1); such a polygon always has an axis of symmetry p; the vertices (poles) N and N' are located symmetricly relatively to the axis p and the octahedron follows ''in whole'' the flexions of the tetrahedral angle $NABCD$. But all Bricard's flexible octahedra have selfintersections. Again only after a long period of time, in 1977, R. Connelly [6] constructed an embedded flexible polyhedron. It was not an easy success even psychologically because just before it H. Gluck [12] proved that (in some sense) almost all topologically sphere-type polyhedra are rigid. Immediately after the Connelly's flexible polyhedron some new examples were built, the simplest one is due to K. Steffen (1978) development of which can be found in many works, e.g., in [8], [7], and in [9] with a detailed description of this and other flexible models.

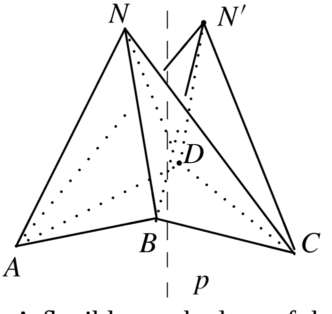

Figure 1: A flexible octahedron of the 1-st type.

It turned out that all these polyhedra possess the same property: their volumes remain invariant during the flexion! This observation gave a reason to R. Connelly to formulate in his talk on Helsinki International Congress of Mathematicians [10] a conjecture that this property is common for all flexible polyhedra and, by obvious reasons, he called it ''Bellows Conjecture'' [2]. In my talk I'll present a proof of ''Bellows Conjecture''. It turns out that it is an easy corollary of a more fundamental result related to volumes of polyhedra which has other interesting consequences too.

convex polygons with the same gap that appeared later in Cauchy's proof too.

[2] In reality I don't know who is the author of the conjecture; Connelly himself told me that he is responsable only for the title ''Bellows Conjecture''. Also in 1978 N. Kuiper formulated the same question in his talk on Bourbaki seminar [11].

2 Two observations about the volume of a polyhedron

Let K be a given geometrical 2-dimensional symplicial complex (that is its simplices are euclidean points, segments and triangles) which triangulates an orientable manifold. We call a mapping $P : K \to R^3$ *polyhedron with combinatorial structure* K (of course, it is supposed that the mapping is continuous on K and linear on its simplices); sometimes we call a polyhedron the image $P(K)$; images of 0-, 1- and 2-simplices of K are called respectively vertices, edges and faces of the polyhedron P. Because the faces are triangles any mapping $P : K \to R^3$ is defined by its values in the vertices of P.

This definition of a polyhedron can give very complicated shapes of its image in three-space, e.g., any selfintersections and degeneracies of faces and edges are admitted. So we need to introduce a specified definition of the volume of such a polyhedron. Any considered polyhedron P is an image of an orientable manifold M; we transfer a choosed orientation of M to P so that any face of P has a suitable orientation. Let O be a point in R^3. We consider now tetrahedra with O as their common vertex and for which the bases are oriented faces of P. The sum of oriented volumes of all such tetrahedra is called the *generalized oriented volume* of P. It is easy to show that this definition is correct in the sense that it does not depend on the choice of the point O. Further if the polyhedron P is embedded then its oriented generalized volume coincides with its ordinary oriented volume.

If n is the number of vertices of P in R^3 then we can associate to P a point M in R^{3n} with the coordinates $(x_1, \dots, z_n)$, where $(x_i, y_i, z_i), 1 \le i \le n$, are coordinates of vertices of P, numbered in some fixed order. Conversely to any point $M(x_1, x_2, x_3; \dots; x_{3n-1}, x_{3n-2}, x_{3n}) \in R^{3n}$ we can associate by the evidently way a polyhedron if we know in advance its combinatorial structure K. So we have a bijective mapping between all polyhedra with a given combinatorial structure and all points in R^{3n}.

The number of edges of P is $e = 3n - 6 + 6g$ where g is the topological genus of K. Let the edges be enumerated by an index $k = k(i, j)$, $1 \le k \le e$, where i, j are indices of vertices joined by the edge numbered k. The lengths l of edges are given by equations

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = l_k^2, \; 1 \le k \le e. \tag{1}$$

If we consider *all* solutions $(x_1, \dots, z_n)$ of (1) under the condition that the right sides, i.e., the lengths l_k, are fixed, we obtain all polyhedra in R^3 with rigid faces *isometric* to P and having the same combinatorial structure K as P has. Now we want to exclude polyhedra obtained by a parallel translation of P; for this we add to system (1) three equations

$$\sum_i x_i = 0, \quad \sum_i y_i = 0, \quad \sum_i z_i = 0. \tag{2}$$

It is easy to show that solutions of (1)–(2) are situated in R^{3n} in a ball B of some finite radius that is $x_1^2 + \cdots + z_n^2 < r^2$ where r depends on the combinatorial structure K and the set $l = (l_1^2, \ldots, l_e^2)$. So the set $\tilde{P}$ of polyhedra in R^3, which have the same combinatorial structure K and isometric to P, is in a homeomorphic correspondence with the algebraic variety $\tilde{A}$ defined by system (1)–(2). But $\tilde{A}$ can have only a finite number of compact connected components in the ball B. If we add to (1)–(2) three new equations excluding continuous rotations[3] of P then any one-point component of the correspondingly changed $\tilde{A}$ corresponds to a rigid polyhedron from $\tilde{P}$ and the others are composed by flexible polyhedra from $\tilde{P}$. Hence we have the first observation: *if the Bellows Conjecture is true then the generalized volumes of all polyhedra from $\tilde{P}$ can have only a finite number of possible values*.

Now we pass to the second observation. Let us remind a formule for the volume of a tetrahedron as a function of the lengths of its edges:

$$\begin{aligned} V^2 \;=\; \frac{1}{144}\,\big[& l_1^2 l_5^2\big(l_2^2 + l_3^2 + l_4^2 + l_6^2 - l_1^2 - l_5^2\big) \\ +\;& l_2^2 l_6^2\big(l_1^2 + l_3^2 + l_4^2 + l_5^2 - l_2^2 - l_6^2\big) \\ +\;& l_3^2 l_4^2\big(l_1^2 + l_2^2 + l_5^2 + l_6^2 - l_3^2 - l_4^2\big) \\ -\;& l_1^2 l_2^2 l_4^2 - l_2^2 l_3^2 l_5^2 - l_1^2 l_3^2 l_6^2 - l_4^2 l_5^2 l_6^2\big], \end{aligned} \tag{3}$$

where $l_1, \ldots, l_6$ are the lengths of the edges (Fig. 2).

Let's consider now a polyhedron P with 5 vertices which has only triangular faces. Then P is combinatorially equivalent to the model drawn in Fig. 3. For the oriented volume V of P we have

$$V = V_1 + \epsilon V_2,$$

where V_1 and V_2 are respectively volumes of tetrahedra $Ap_1p_2p_3$ and $Ap_1p_3p_4$ and $\epsilon = \pm 1$. In this equation we can eliminate ϵ and we obtain that V satisfies the equation

$$V^4 - 2\big(V_1^2 + V_2^2\big)V^2 + \big(V_1^2 - V_2^2\big)^2 = 0. \tag{4}$$

[3] By the way it is not known how to write those equations in such a way that they are valid for *all* polyhedra with a given combinatorial structure K as this was in the case of excluding parallel translations by equations (2); for a discussion of this problem see [13].

But V_1^2 and V_2^2 are expressed by (3) as functions on the lengths of the edges of P and we can formulate our second observation:

> *In the simplest cases the volume of a polyhedron is a root of some polynomial equation whose coefficients depend only on the lengths of the edges of the polyhedron.*

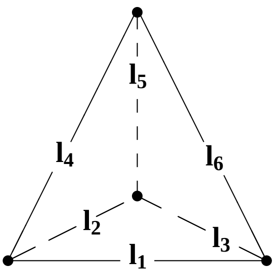

Figure 2: A tetrahedron.

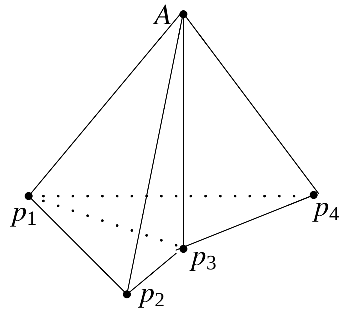

Figure 3: A 5-vertices polyhedron.

3 A generalization of Heron's formula to the volume of a polyhedron

These two observations lead us to an idea that for the validity of the Bellows Conjecture it is sufficient to prove that for any polyhedron P there exists a polynomial or even an analytic equation defined only by the combinatorial structure and the metric of P such that the generalized volumes of P and of all isometric to P polyhedra are roots of this equation. It turns out that this new conjecture is true. Namely one can prove the following generalization of the Heron's formula to volumes of polyhedra.

Theorem 1. *Let $\tilde{P}$ be the family of all polyhedra in R^3 which have a fixed combinatorial structure K and fixed values of the edge lengths l_k, $1 \le k \le e$, where e is the number of edges (which is, of course, the same for all polyhedra in $\tilde{P}$). Then there exists a volume polynomial for $\tilde{P}$, that is, a polynomial*

$$Q(l, V) = V^{2N} + a_1(l)V^{2N-2} + \cdots + a_{N-1}(l)V^2 + a_N(l), \qquad (5)$$

such that the generalized volume of any polyhedron in $\tilde{P}$ is one of its roots. Furthermore, the coefficients a_i are polynomials in $l = (l_1^2, \dots, l_e^2)$ with rational coefficients depending on K.

A sufficiently detailed proof of the theorem can be found in [13], [14] and [15], a sketch of proof is in [16].

Corollary 1. *The Bellows Conjecture is true.*

Indeed the volume of a flexible polyhedron has to be in a continuous dependence on the flexion and simultaneously it can take only a finite number of values as a root of a fixed polynomial equation; so it is a constant.

4 Some additional information about polynomial equations for the volume

1) *An algebraic sense of Theorem 1.* Equations (1) say that the squares of the lengths of the edges of any polyhedron with n vertices are polynomials of the second degree in coordinates $(x_1, \dots, z_n)$. This fact we will represent by the expression $l = l(x)$. On the other hand the generalized volume can be presented as a polynomial in coordinates $(x_1, \dots, z_n)$ too. For this we have to use the representation of the volume of a tetrahedron by the mixed product as a function of coordinates of vectors composing this tetrahedron. Then the generalised volume, as the sum of mixed products, will be a polynomial of degree 3, and this fact we will represent by the formula $V = V(x)$. Now, if we substitute the polynomial expressions for the volume and lengths in the volume polynomial $Q(l, V)$ in (5) then we will have $Q(l(x), V(x)) \equiv 0$ for all $(x_1, \dots, z_n)$. In other words the equation (5) means that *the functions $l(x)$ and $V(x)$ are algebraically dependent.*

2) Besides proofs mentioned above there is an other proof of theorem 1, see [17]. The formulation of the assertion proved in [17] is more general. Namely, let L be a field (e.g., the field $\mathbf{R}$), $p_1 = (x_1, y_1, z_1), \dots, p_n = (x_n, y_n, z_n)$ be images of 0-simplices of an abstract simplicial complexe K under an mapping $P : K \to L^3$ (it is supposed that K triangulates an orientable manifold and n is the number of 0-simplices of K). The mapping P defines a polyhedron or a polyhedral surface $P(K)$. Let the generalized volume V of the polyhedron $P(K)$ be defined as a polynomial in L by the following formula

$$V = \mathrm{vol}\,(P) = \frac{1}{6} \sum_{[p_i, p_j, p_k] \in F_+} \det\,[p_i, p_j, p_k],$$

where the summation is taken over all positively oriented faces F_+ of P. Further, let $R \subset L$ be a ring generated by elements $(p_i - p_j)^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \in L$, where p_i and p_j are images of all those 0-simplices of K which compose 1-simplices of K (in other words p_i and p_j are those vertices of P which are joined by an edge). Then the theorem in [17] asserts that $v = 12V$, as an element of L, is integer over the ring R; this means that v satisfies a polynomial equation

$$v^N + a_1 v^{N-1} + \cdots + a_{N-1} v + a_N = 0,$$

where all coefficients a_i belong to R.

3) The proof in [17] is shorter than other ones but it is not constructive. Contrarily, proofs in [14]–[15] are constructive and they permit to find a lot of such polynomials which are annuled by the volumes of isometric polyhedra with the same combinatorial structure. By this reason it is interesting to find a *canonical* volume polynomial that is one having the smallest degree in V. This is made for polyhedra homeomorphe to the sphera. In [18] the following theorem is proven:

Theorem 2. *Let S be the set of all polynomials Q(l,V) for polyhedra with a combinatorial structure K of genus* $g = 0$*, such that* $Q(l(x), V(x)) \equiv 0$*. Let d be the smallest (non zero) degree in V of the polynomials in S. Then among the polynomials of degree d in S there is a unique monic (that is whose leader coefficient is 1) polynomial* Q_0 *which divides all polynomials in S.*

Theoretically the procedure of application of Theorem 2 is as follows: we can construct at least one polynomial from the set S by the proof of Theorem 1, so we have to decompose it in factors with polynomial coefficients and one of those factors has to be the canonical volume polynomial Q_0. But in reality this procedure is not applicable (at least for the present) because the demanded quantity of calculations is too big. We found such a canonical volume polynomial for octahedra by some other method. It is of degree 16 and contains many millions of monoms! (Compare this with polynomial (1) for a tetrahedron containing 23 monoms and this one (4) for polyhedra with 5 vertices which has approximately already 1000 terms.) In practice for finding the canonical volume polynomial it is sufficient to replace lengths l by their given or known values and a special programme writes the polynomial with numerical coefficients, see [14]. In the case when some edges of a polyhedron have equal lengths (e.g. by reason of a symmetry) the canonical volume polyhedron becomes more compact and one can hope to have for it a sufficiently small expression, see [19] and [20].

In the case of polyhedra of an arbitrary genus $g > 0$, the problem of finding and uniqueness of the canonical volume polynomial is an open question.

4) Volume polynomials have to reflect some geometrical properties of those polyhedra for which they are constructed. For example one can show that if a considered polyhedron is flexible then it has a volume which is a multiple root of the volume polynomial constructed in the proof of Theorem 1, [21]. We conjecture that this property is common for all volume polynomials including canonical ones. Moreover the order of multiplicity of a root has to be related with some characteristics of discrete isometric transformations of the polyhedron under consideration.

Further, the volume polynomial has to reflect some symmetry properties of the polyhedron via some properties of Galois group of the polynomial.

5) *A necessary condition for isometric realizability of a given polyhedral metric*. For any polyhedron a volume polynomial $Q(V)$ can be composed using only its combinatorial structure K and the lengths of its edges, that is, we don't need to know the polyhedron itself. By this reason, if a development D of triangles is given for which we know that topologically it is homeomorphic to an orientable manifold then we can construct a volume polynomial $Q(V)$ for volumes of polyhedra whose faces are congruent to corresponding triangles of D. Thus, we did not yet find a polyhedron isometric to the development D in the indicated sense but we know already that the volume of that polyhedron must be among roots of the constructed polynomial $Q(V)$. From this we deduce: if all roots V^2 of $Q(V)$ are negative or complex then the given development is not isometrically realizable in R^3 as a polyhedron with given faces. So *existence of at least one nonnegative root of $Q(V)$ is a necessary condition for the isometric realizability* of the given development D.

6) Note that Theorem 1 is proven only in the 3-dimensioanal Euclidean space. In the spherical space it is not true [22]. For other cases, namely in the hyperbolic space as well as in multidimensional Euclidean spaces the question is still open.

7) Recently J.-M.Schlenker presented at the Bourbaki seminar a beautiful survey on the topic [23].

5 Polyhedra with given volumes

Let $Q(V)$ be a volume polynomial constructed using only a given development D of triangles. Let $V_0^2 \geq 0$ be a root of this polynomial. The question is: *can we affirm that this root is the square of the volume of a polyhedron isometric to D*? If the answer is *''yes''* we will say that the root V_0 is *realisable* as the volume. This question is far to be solved and some first results obtained in [24] show that the situation has to be very complicated. Now we can work closely only with some special octahedra for which the polynomial $Q(V)$ can be written explicitly. Namely let K be a metrical simplicial complex, that is, to any 1-dimensional simplex of K a positive number is prescribed considered as its length. We know that we can compose a polynomial equation

$$Q(V) = V^{2N} + a_1(l)V^{2N-2} + \cdots + a_N(l) = 0,$$

such that the volume of any polyhedron $P(K)$ isometric to K is a root of this equation. In the general case even for an octahedron this equation is too complicated to be written explicitly. But if an octahedron has a symmetry then some of its 12 edges have the same values of lengths and in this case the coefficients $a_i(l)$ may be presented explicitly on the paper. Let an octahedron be modelled as on the Fig. 4. Let's denote K an octahedron with the following lengths of edges

$$|A_1B_1|^2 = |A_2B_2|^2 = a, \quad |A_1B_2|^2 = |A_2B_1|^2 = b, \quad |B_1C_1|^2 = |B_2C_2|^2 = c,$$
$$|B_1C_2|^2 = |B_2C_1|^2 = d, \quad |A_1C_1|^2 = |A_2C_2|^2 = e, \quad |A_1C_2|^2 = |A_2C_1|^2 = f.$$

Then the polynomial equation for the volume is as follows

$$\begin{aligned} Q(V) \; = \; & V^{16} - 4\big[ab(c+d+e+f-a-b) \\ & + \; cd(a+b+e+f-c-d) \; + \; ef(a+b+c+d-e-f) \\ & - \; (ace+adf+bcf+bde)\big]\, V^{14} \; = \; 0, \end{aligned}$$

where $V = 6v = 6 \;\text{vol}\; P(K)$.

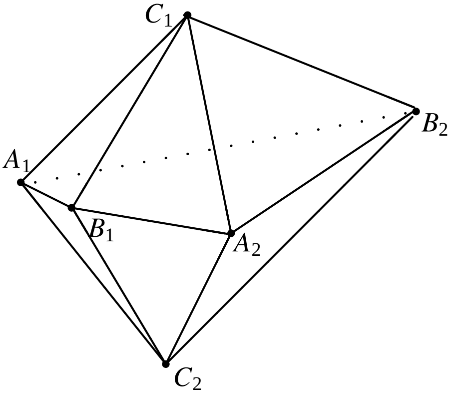

Figure 4: An octahedral model.

Theorem 3 ([24]). *If the root*

$$\begin{aligned} V^2 \; = \; & 4\big[ab(c+d+e+f-a-b) + cd(a+b+e+f-c-d) \\ & + \; ef(a+b+c+d-e-f) \; - \; (ace+adf+bcf+bde)\big] \end{aligned}$$

of the equation $Q(V) = 0$ is positive then $v(= V/6)$ can be realised as the volume of an octahedron isometric to K with the given above lengths of the edges.

Let's consider a model of K with the following lengths of the edges:

$$|A_1B_1|^2 = |A_2B_2|^2 = a, \;\; |A_1B_2|^2 = |A_2B_1|^2 = b, \;\; |B_1C_1|^2 = |B_2C_1|^2 = c,$$

$$|B_1C_2|^2 = |B_2C_2|^2 = d, \;\; |A_1C_1|^2 = |A_2C_1|^2 = e, \;\; |A_1C_2|^2 = |A_2C_2|^2 = f.$$

In this case the polynomial equation is of form

$$Q(V) \;=\; V^{16} + p(l)V^{14} \;+\; q(l)V^{12} \;=\; 0, \quad V \;=\; 6v \;=\; 6\,\mathrm{vol}\,(P).$$

After the reduction we have an equation $Q_1(V) = V^4 + pV^2 + q = 0$.

Theorem 4 ([24]). *1) If the equation $Q_1(V)$ has only one positive root V^2 then $v = V/6$ is the volume of a really existing octahedron $P(K)$ isometric to K. 2) If the equation has two different positive roots V_1^2 and V_2^2 then the both $v_1 = V_1/6$ and $v_2 = V_2/6$ are volumes of really existing octahedra isometric to K. 3) If the positive root of the equation is multiple (that is $V_1^2 = V_2^2 > 0$) then $v_1 = V_1/6$ is the volume of a really existing octahedron under the additional condition $-ab + cd - ed - cf + ef \neq 0$, otherwise the existence of such an octahedron is not obligatory.*

6 Algorithmic solution of the problem of isometric realization of a given development of triangles

To construct a polyhedron from a given development of triangles as its ''material'' it is sufficient to find dihedral angles between its faces. or to know the lengths of diagonals joining two not common vertices of two faces with a common edge; we will refer these diagonals as *small* ones. It turns out that, for any small diagonal of a polyhedron which is generic in some sense, one can prove the following theorem [25]:

Theorem 5. *For every small diagonal there is a polynomial equation whose coefficients depend only on the metric and the combinatorial structure of the polyhedron, and, for polyhedra in a general position, not all these coefficients are zero.*

Thus the set of possible values of lengths of small diagonals is finite and we can test all possible isometric realizations in a finite number of steps.

Now we see that we can find any metric characteristic of a polyhedron if we know its combinatorial structure and metric. We call this kind of calculus ''solution of polyhedra'' by analogy with well known term ''solution of triangles''. And we can say that the metric theory of polyhedra becomes a finite science at least in the same sense as chess is a finite play.

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